PD Modelling: a Bayesian-Logit approach - Sovereign Focus

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Some disclaimer

All views expressed in this presentation are my own and do not represent the opinions of any entity whatsoever with which I have been, am now, or will be affilated.

Modelling alternatives

- Scoring model
  - Probability of Default determined by a set of explicative variables
  - Credit risk scorecard
  - Multivariate logistic regression or other supervised learning models
- ODF model
  - Probability of default is a direct function of the Observed Default Frequency
  - Univariate *logit* regression
- Market risk model
  - Bond price
  - CDS spread

Determinants of model selection

Model determinants:

- Data availability
- Default frequency
- Analytical skills
- Budget

$\Rightarrow$

Model properties:

- *Correct*
- *Consistent*
- *Stable*

Sovereign modelling peculiarities

- Small samples
- Very low default frequency
- Asymmetrical distribution of defaults (skewed towards lower rating grades)
- Counterparty specific characteristics:
  - Local governments
  - Government entities
  - Central government with exposure denominated in local currency*
  - Central government with exposure denominated in foreign currency

Sovereign modelling and fiat currencies

Fiat currency: a **floating non-convertible** currency (US Dollar, British Pound, Swedish Crown).

A country that issues its own fiat currency does not have any financial constraints for its expenditures/investments in its own currency.
Unlike households, firms or states with non-fiat currency (e.g. a state in the Eurozone) which are *money users*, a fiat-state is a *money issuer* hence it can always afford to honour its liabilities, no matter how big they become.

Probability of Default should be set to **0** if **all** these conditions apply:

- Investment grade credit quality
- Exposure to a central government
- Exposure denominated in the currency of the central government
- Currency issued by the central government is *fiat*

More about currency regimes and solvency:
`https://onexcent.com/heterodox-macroeconomics/`

Model proposals

- Logit regression
- Bayes-Logit regression
- Constrained Bayes-Logit regression

Walkthrough in **R** $\rightarrow$

R code available on:
`https://onexcent.com/banking-and-finance/`

Logit regression

Dependent variable $\rightarrow$ *logit* of the observed default frequency per risk grade (θ^g).
Independent variable $\rightarrow$ I^g = grade scale index (grade ID).

$$\ln\left(\frac{\theta^g}{1-\theta^g}\right) = \beta_0 + \beta_1 I^g + \epsilon \qquad (\textit{log-linear model}) \tag{1}$$

The estimated parameters $\hat{\beta}_0$ and $\hat{\beta}_1$ are used to obtain the *logit PD*.

$$\widetilde{PD^g} = \hat{\beta}_0 + \hat{\beta}_1 I^g \tag{2}$$

The exponential *PD* is obtained through *inverse logit transformation*.

$$PD^g = \frac{1}{1+\exp^{-\widetilde{PD^g}}} \tag{3}$$

Bayes-Logit regression I

Based on Orth (2012) and Kleinman (1973)

Introduction:
Empirical Bayes method used to determine the a-priori and a-posteriori distribution of the default rate by *combining* data from other portfolios.

Assumptions:
Beta-Binomial framework for the default rate.
For each portfolio p ($p = 1, ..., P$) the a-priori default rate is Beta distributed.

$$\theta^{g,p} \sim \mathrm{Beta}(\alpha, \beta) \tag{4}$$

And the number of default is Binomial.

$$D^{g,p} | \theta^{g,p} \sim \mathrm{Binom}(\widetilde{N}^{g,p}) \tag{5}$$

Bayes-Logit regression II

Prior mean for $\theta^{g,p}$ $\rightarrow$ Portfolio-weighted average default frequency

$$\hat{\mu^g} = \Sigma_{p=1}^{P} \hat{w}^{g,p} \hat{\theta}^{g,p} \tag{6}$$

Prior standard deviation for $\theta^{g,p}$ $\rightarrow$ τ^g formula by Kleinman (1973)

$$\hat{\tau}^g = \frac{\frac{P-1}{P} \Sigma_{p=1}^{P} \hat{w}^{g,p} (\hat{\theta}^{g,p} - \hat{\mu}^g)^2 - \hat{\mu}^g (1 - \hat{\mu}^g) \left(\Sigma_{p=1}^{P} \frac{\hat{w}^{g,p}(1-\hat{w}^{g,p})}{N^{g,p}} \right)}{\hat{\mu}^g (1 - \hat{\mu}^g) \left(\Sigma_{p=1}^{P} \frac{N^{g,p}-1}{N^{g,p}} \hat{w}^{g,p} (1 - \hat{w}^{g,p}) \right)} \tag{7}$$

Where:

$N^{g,p}$ = number of observations per grade and portfolio

$D^{g,p}$ = number of defaults per grade and portfolio

$\hat{\theta}^{g,p} = \frac{D^{g,p}}{N^{g,p}}$ $\rightarrow$ ODF per grade and portfolio

$\hat{w}^{g,p} = 1/P$ or $N^{g,p} / \sum_{p=1}^{P} N^{g,p}$

Bayes-Logit regression III

Empirical Bayes estimator for the posterior mean of $\theta^{g,p}$:

$$\hat{\theta}_{EB}^{g,p} = \frac{1 - \hat{\tau}^{g}}{1 + \hat{\tau}^{g}(N^{g,p} - 1)}\hat{\mu}^{g} + \frac{\hat{\tau}^{g} N^{g,p}}{1 + \hat{\tau}^{g}(N^{g,p} - 1)}\hat{\theta}^{g,p} \qquad (8)$$

The Empirical Bayes estimator formula can be interpreted as a **weighted average** of the **prior mean** ($\hat{\mu}^{g}$) and the **observed default frequency** estimate for portfolio p.

Bayes-Logit regression IV

Regression model:

Dependent variable $\rightarrow$ *logit* of the empirical-Bayes estimated ODF of the Sovereign portfolio ($\hat{\theta}^{g,p}_{EB},\ where\ p = Sovereign$).
Independent variable $\rightarrow$ grade scale index (I^g).

$$\ln\left(\frac{\hat{\theta}^g_{EB}}{1-\hat{\theta}^g_{EB}}\right) = \beta_0 + \beta_1 I^g + \epsilon \qquad (\textit{log-linear model}) \qquad (9)$$

PD estimates are obtained like in the *simple Logit*.

$$\widetilde{PD^g_{EB}} = \hat{\beta}_0 + \hat{\beta}_1 I^g; \qquad PD^g_{EB} = \frac{1}{1+\exp^{-\widetilde{PD^g_{EB}}}}$$

Constrained Bayes-Logit regression

Dependent and independent variable $\rightarrow$ same as the (unconstrained) Bayes-Logit model.

Regression parameters β_0 and β_1 are optimised under a conservatism constraint.

log-linear model: $$\ln\left(\frac{\hat{\theta}^g_{EB}}{1-\hat{\theta}^g_{EB}}\right) = \beta_0 + \beta_1 I^g + \epsilon$$

PD estimates: $$PD^g_{EB} = \frac{1}{1+\exp^{-(\hat{\beta}_0+\hat{\beta}_1 I^g)}}$$

constraint: $$PD^g_{EB} \geq \hat{\theta}^g_{EB}(k), \quad \forall\, g = 1, ..., G$$

Where the constant k is used to control the degree of conservatism (severity of the constraint).

k $= 0 \rightarrow$ no conservatism
k $> 0 \rightarrow$ more conservatism
k $< 0 \rightarrow$ less conservatism